

Mathematical Physics: Lecture 1 — Vector Spaces

Giovanny Espitia

- A **vector space** V is a set such that $\bar{v} \in V$.
- Two operations are defined on V : addition, and multiplication by scalar quantities.
- The result of either operation must remain in V (closure).
- The coefficients (scalars) can be complex.
- Addition and multiplication are associative and commutative:

$$\bar{u} + \bar{v} = \bar{v} + \bar{u} \quad (1)$$

$$\alpha(\beta\bar{v}) = (\alpha\beta)\bar{v} \quad (2)$$

$$\alpha(\bar{u} + \bar{v}) = \alpha\bar{v} + \alpha\bar{u} \quad (3)$$

$$(\alpha + \beta)\bar{v} = \alpha\bar{v} + \beta\bar{v} \quad (4)$$

For every $\bar{u} \in V$ there is an additive inverse:

$$\bar{u} + \bar{v} = \bar{0} \quad (5)$$

$$\bar{u} + \bar{0} = \bar{u} \quad (6)$$

$$-\bar{u} = (-1)\bar{u} \quad (7)$$

$$0 \cdot \bar{u} = \bar{0} \quad (8)$$

where $\bar{0}$ denotes the **null vector**.

- A **Hilbert space** is a class encompassing many kinds of vector spaces.
- Another example is $\mathbb{R}^3$.

Examples of Vector Spaces

- Polynomials of degree p form a vector space ($p = 2$ is the quadratic).
- Vector spaces satisfy the properties of vector spaces listed above.
- Function spaces are also vector spaces.
- Directional derivatives (relevant in GR — should be *directional* derivatives). Directional derivatives are proper representations of vector spaces.
- Spherical harmonics, Lagrange polynomials, Chebyshev polynomials, etc. are all members of vector spaces.

Linear Independence and Dependence

$$\sum_i \alpha_i \bar{v}_i = 0 \quad (9)$$

If $\alpha_i = 0$ (for all i) is the only solution, the set $\{\bar{v}_i\}$ is **linearly independent**. Otherwise, the vectors are **linearly dependent**.

How to prove linear dependence/independence: $\Rightarrow$ no basis vector can be written as a sum of the other vectors.

- Basis vectors do not have to be normalized nor orthogonal. (These are simply nice properties to have.)
- Is $\{1, x, x^2\}$ a linearly independent basis?
- $\dim V \Rightarrow$ the number of vectors in a basis (equivalently, the number of linearly independent vectors).
- Scalars $\Rightarrow$ components of a vector (in a given basis); scalars multiply vectors and can themselves be functions, etc.

Combining Vector Spaces

Discussion Question

Can we only add members of the same vector space? Is any set of linearly independent vectors a basis?

$$\bar{u} + \bar{v} = \sum_i u_i \bar{e}_i + \sum_i v_i \bar{e}_i = \sum_i \omega_i \bar{e}_i \quad (10)$$

$$\sum_i (u_i + v_i) \bar{e}_i = \sum_i \omega_i \bar{e}_i \implies \omega_i = u_i + v_i \quad (11)$$

Discussion Question

Why can't we add members of different vector spaces? Can we form a more general vector space?

- Once a basis is fixed, we work with components.
- We *can* add elements of different vector spaces by mapping from one vector space to another:

$$f : \bar{V} \rightarrow \bar{U}, \quad g : \bar{U} \rightarrow \bar{V} \quad (12)$$

For example, we can map $\mathbb{R}^3 \rightarrow \mathbb{R}^2$.

Discussion Question

Is Fock space a space of many particles?

Change of Basis

- We can change basis via rotations. This is akin to applying a unitary operator on states, which conserves the norm.
- Observables might be related to one another via a unitary transformation.
- Indices should appear at most twice.

$$\bar{e}_j = S_{i'j} \bar{e}_{i'} \quad (13)$$

$$\bar{u} = u_i \bar{e}_i = u_i S_{j'i} \bar{e}_{j'} \quad (14)$$

$$u_{j'} = S_{j'i} u_i \quad (15)$$

$S_{ij} e_j \Rightarrow$ matrix. We can make comparisons:

$$u_i = (S^{-1})_{ij} u_j \quad (16)$$

$$\bar{e}_{j'} = (S^{-1})_{ij'} \bar{e}_i \quad (17)$$

Not all basis transformations are linear. For example, spherical coordinates connect to Cartesian coordinates nonlinearly. The Kronecker delta $\Rightarrow$ a linear transformation.

$$u_i = (S^{-1})_{ij} u_j \quad u_{j'} = S_{j'i} u_i \quad (18)$$

$$u' = S u \quad u = S^{-1} u' \quad (19)$$

Rotations are linear transformations:

$$\boxed{S_{i'j} = \frac{\partial x_{i'}}{\partial x_j}} \longrightarrow \text{linear transformation.} \quad (20)$$