

Mathematical Physics: Lecture 2 — Inner Products and Linear Operators

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Only linear transformations can be written as matrix products. A linear transformation is essentially a map from one space to another.

The Inner (Scalar) Product

We equip our vector space with an inner product.

$$\langle \cdot | \cdot \rangle : V \times V \rightarrow \mathbb{C}, \quad \langle \bar{u} | \bar{v} \rangle = c \quad (1)$$

i.e. a map from $V \times V \rightarrow \mathbb{C}$.

Linearity property:

$$\langle u + v | w \rangle = \langle u | w \rangle + \langle v | w \rangle \quad (2)$$

$$\langle u | av \rangle = a \langle u | v \rangle \quad (3)$$

If the inner product maps to $\mathbb{R}$, the dot product is symmetric in its arguments:

$$\langle \bar{v} | a\bar{u} \rangle = a \langle \bar{v} | \bar{u} \rangle, \quad \langle av | u \rangle = a^* \langle v | u \rangle \quad (4)$$

Example ($\mathbb{R}^3$):

$$\bar{u} \cdot \bar{v} = \langle \bar{u} | \bar{v} \rangle \quad (5)$$

- Used to define the norm of a vector: $\|u\| = (\langle u | u \rangle)^{1/2}$.
- This should be positive definite.
- The scalar product is related to the angle between vectors: $\bar{A} \cdot \bar{B} = \|\bar{A}\| \|\bar{B}\| \cos \theta$.

$$\bar{u} \cdot \bar{u} = \sum_i u_i^2 \geq 0, \quad \text{equal to 0 only if } \bar{u} = \bar{0}. \quad (6)$$

- Two functions or vectors are **orthogonal** if their inner product is zero.
- The number of basis vectors in a space is the dimension of the space — what is the rank of a matrix?

Orthogonal basis:

$$\langle \bar{e}_i | \bar{e}_j \rangle = \begin{cases} 0, & i \neq j \\ \langle \bar{e}_i | \bar{e}_i \rangle, & i = j \end{cases} \quad (7)$$

If we have an orthonormal basis:

$$\langle \hat{e}_i | \hat{e}_j \rangle = \delta_{ij} \quad (8)$$

$$\bar{u} = u_i \hat{e}_i, \quad \bar{v} = v_j \hat{e}_j \quad (9)$$

$$\langle \bar{u} | \bar{v} \rangle = \langle u_i \hat{e}_i | v_j \hat{e}_j \rangle = u_i^* v_j \langle \hat{e}_i | \hat{e}_j \rangle = u_i^* v_j \delta_{ij} = u_i^* v_i = (\bar{u}^*)^T \bar{v} \quad (10)$$

$$A = A_{ij}, \quad A^T = A_{ji}, \quad A^\dagger = (A^*)^T \quad (11)$$

This can be used to get components by projecting onto a basis. Let $\bar{u} = u_i \hat{e}_i$; to obtain u_2 :

$$\langle \hat{e}_2 | \bar{u} \rangle = u_i \langle \hat{e}_2 | \hat{e}_i \rangle = \delta_{2i} u_i = u_2 \quad (12)$$

Linear Operators

$A : V \rightarrow W$ is a map from one vector space to another vector space.

$$A(\bar{u}) = \bar{v} \quad (13)$$

Linearity property: $A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v}$

Distributive property: $(A + B)\bar{u} = A\bar{u} + B\bar{u}$

Discussion Question

What are the requirements for a matrix to have an inverse?

Non-singular, i.e. $\det \neq 0$.

If two operators commute, they can be *simultaneously measured*!

$$A(\bar{u}) = A(u_i \hat{e}_i) = u_i A(\hat{e}_i) = u_i \sum_j A_{ji} \hat{e}_j \quad (14)$$

$$\langle \hat{e}_k | A(\hat{e}_j) \rangle = \sum_i \langle \hat{e}_k | A_{ij} | \hat{e}_i \rangle = \sum_i A_{ij} \delta_{ki} = A_{kj} \quad \text{matrix element: projection of operator onto basis} \quad (15)$$

$$\bar{v} = A\bar{u} \implies v_i = A_{ij} u_j \quad (\hat{e}_i = \delta_{ij} e_j) \quad (16)$$

Invertibility, Subspaces, and Determinants

- The null vector always exists (i.e., $A\bar{0} = \bar{0}$).
- If a linear transformation results in a basis vector becoming zero, then that is a map *onto a subspace*, and therefore no longer has an inverse.
- If a linear transformation projects onto a subspace, then we can assume the transform is represented by a *singular* matrix ($\det A = 0$).

- If $\det A = 0$, then A^{-1} does not exist.
- $\det A \rightarrow$ volume change.
- If A projects onto a subspace, then a dimension is contracted, and the new volume is zero.

If a linear transformation maps onto a subspace, then a dimension is contracted. Thus $\det A = 0$.

- Determinants are volume changes.
- Matrices are basis dependent.

$$\hat{K} = S S^{-1} \quad (17)$$

Similarity Transformations

$$A = S^{-1} A' S \quad (18)$$

$$A' = S A S^{-1} \quad \longrightarrow \text{basis transformation. Express } A' \text{ in the } A \text{ basis.} \quad (19)$$

$$A'_{i'j'} = \sum_{k,m} S_{i'k} A_{km} S_{mj'}^{-1} \quad (20)$$

A similarity transform is a basis transform for operators; this is only valid when the inner product is over $\mathbb{R}$ (or more generally, when the basis change is unitary).

$$A_{i'j'} = \langle \hat{e}_{i'} | A \hat{e}_{j'} \rangle \quad (21)$$

We can then get the matrix on a component-by-component basis.

SCALAR PRODUCTS ARE BASIS INDEPENDENT!