

Mathematical Physics: Lecture 3 — Eigensystems

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So far, we have discussed chapters 2, 3, 5.

$$\text{Vector: } \bar{u} \in V, \quad \text{basis: } \bar{e}_i, \quad \bar{u} = \sum_i u_i \bar{e}_i \quad (1)$$

To take the dot product of elements expressed in a different basis, we must first map them to the same vector space.

$$\text{Linear operator: } A : V \rightarrow V, \quad \bar{v} = A\bar{u}, \quad v_j = \sum_k A_{jk} u_k \quad (2)$$

- Inner products are positive definite.
- We transform a linear operator via a similarity transform. We know S exists, since we can go between bases via S , and thus S^{-1} backwards.

$$\bar{e}_{\bar{k}} = S_{i\bar{k}} \bar{e}_i, \quad \bar{e}_i = S_{\bar{k}i}^{-1} \bar{e}_{\bar{k}} \quad (3)$$

- Eigensystems (Chapters 3 and 6): $A\bar{v} = \lambda\bar{v}$
- How to get matrix elements using inner products, and having to project out components via the inner product with the proper basis vector.

Chapter 3: Curls, Divergences, and Related Theorems

So far, we have studied finite-dimensional vector spaces.

Eigenvector picture: for a generic vector $\bar{v}$, the image $A\bar{v}$ points in a different direction than $\bar{v}$. If $\bar{v}$ happens to be an **eigenvector** of A , then $A\bar{v}$ simply scales $\bar{v}$ by some constant λ (no change in direction).

- If $\bar{u}$ is an eigenvector of A , then A simply scales the vector by some constant.
- Refer to the book for examples of eigensystems.
- Fields are described by continuous phenomena. For example, a perturbation in a field is described by an infinite number of degrees of freedom.
- The analog of the eigenvalue equation is the propagator equation, and this in turn needs a Green's function description.

Worked Example

$$A = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}, \quad Av = \lambda v \quad (4)$$

$$\det \begin{pmatrix} 2 - \lambda & 1/2 \\ 1/2 & 2 - \lambda \end{pmatrix} = 0 \quad (5)$$

$$(2 - \lambda)^2 - \frac{1}{4} = 0 \quad (6)$$

$$(2 - \lambda)^2 = \frac{1}{4} \quad (7)$$

$$2 - \lambda = \pm \frac{1}{2} \quad (8)$$

$$\lambda = \frac{3}{2}, \frac{5}{2} \quad (9)$$

- If $\det(\cdot) = 0 \Rightarrow$ the matrix is non-invertible. In this case, we can get a non-trivial solution.

Discussion Question

What is the fundamental theorem of calculus?

- For an $N \times N$ system, we generally have N solutions.
- Eigenvectors are always defined up to some (multiplicative) constant.
- If our linear transformation is real, then the eigenvalues can be real or complex. If real, then $\lambda = \lambda^*$; otherwise, λ and λ^* are both eigenvalues of the matrix.

$$a_N(\lambda^N)^* + a_{N-1}(\lambda^{N-1})^* + \dots = 0 \quad \text{if } A \text{ is real.} \quad (10)$$

Degeneracy and Matrix Types

Repeated solutions: degeneracy.

- If we change basis, the eigenvalues are still eigenvalues of the rotated matrix. This is because $A\bar{v} = \lambda\bar{v}$ is a basis-independent equation. The degeneracies are therefore retained.

Discussion Question

What is a symmetric matrix?

$$\text{Orthogonal: } A^T = A^{-1} \quad (11)$$

$$\text{Symmetric: } A^T = A \quad (12)$$

Proper rotations are orthogonal and symmetric.

Hermitian Operators

Hermitian operators have real eigenvalues.

$$A\bar{v}_n = \lambda_n \bar{v}_n \quad (13)$$

$$\bar{v}_m^\dagger A\bar{v}_n = \lambda_n \bar{v}_m^\dagger \bar{v}_n \quad (14)$$

$$\bar{v}_m^\dagger A^\dagger \bar{v}_n = \lambda_n \bar{v}_m^\dagger \bar{v}_n \quad \text{if } A \text{ is Hermitian, } A = A^\dagger \quad (15)$$

$$(A\bar{v}_m)^\dagger \bar{v}_n = \lambda_n \bar{v}_m^\dagger \bar{v}_n \quad (16)$$

$$(\lambda_m \bar{v}_m)^\dagger \bar{v}_n = \lambda_n \bar{v}_m^\dagger \bar{v}_n \quad (17)$$

$$\lambda_m^* \bar{v}_m^\dagger \bar{v}_n = \lambda_n \bar{v}_m^\dagger \bar{v}_n \quad (18)$$

$$(\lambda_m^* - \lambda_n) \bar{v}_m^\dagger \bar{v}_n = 0 \quad (19)$$

If $\lambda_n = \lambda_m^*$, then these are real.

Which suggests that $\bar{v}_m$ and $\bar{v}_n$ are orthogonal: if $\lambda_m \neq \lambda_n$, then $\bar{v}_m^\dagger \bar{v}_n = 0$, where $\bar{v}_m^\dagger \bar{v}_n \Rightarrow$ the complex inner product.

- From this we learn that Hermitian matrices have an orthogonal/orthonormal basis. And this is a complete basis.
- Hermitian operators form a complete orthonormal basis.
- In its own basis, an operator is diagonal. This is only true if the eigenvectors are orthogonal.
- A matrix is diagonal in its own eigenbasis.

Normal Matrices and Diagonalization

$$\text{Normal matrix} \implies AA^T = A^T A \quad (20)$$

Normal matrices commute with their transpose: $[A, A^T] = 0$.

$$S = [\bar{v}_1 \quad \bar{v}_2 \quad \cdots \quad \bar{v}_N] \longrightarrow \text{this matrix diagonalizes a matrix.} \quad (21)$$

Discussion Question

Is $A = SDS^{-1}$ true only for certain matrices, or is it true for all matrices? Are there matrices that cannot be diagonalized?

$$\det(A) = \det(SDS^{-1}) = \det(S) \det(D) \frac{1}{\det(S)} = \det(D) = \prod_{i=1}^N \lambda_i \quad (22)$$

- The trace is a sum of eigenvalues $\Rightarrow$ the trace is basis independent.
- The determinant is a product of eigenvalues $\Rightarrow$ the determinant is basis independent.

Discussion Question

Can determinants be complex?