

Lecture 4:

Expansion of functions into some basis.

For example $f(x) = x$ $x \in [0, 1]$ and $\phi_n(x)$

$$f(x) = \sum_n b_n \phi_n(x) \quad b_n \propto \frac{1}{n} \quad \text{on a quantity is the value of } f \text{ over orbit.}$$

Periodic functions should have a periodic basis.

- Sine is an odd function and cosine is an even function
- More accuracy, more basis elements
- Sine basis gives the more best

Hilbert space:

Consider vector space equipped with an inner product.

$$\langle f, g \rangle = \int_a^b f(x) g(x) w(x) dx \quad x \in [a, b]$$

With this norm, there exists a complete basis $f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)$

We can use Gram-Schmidt to generate basis one by one Legendre, Hermite, etc.

Resolution of Identity $\hat{1} = \sum_n |\phi_n\rangle \langle \phi_n|$

$\{|\phi_n\rangle\}$ is a complete orthonormal basis.

$$\sum_n |\phi_n\rangle \langle \phi_n| f \rangle = \int_a^b \phi_n^*(x) f(x) w(x) dx \Rightarrow a_n$$

$$= \sum_n a_n \phi_n(x)$$

Dirac Delta: $\delta(x-y)$ $\int_{-\infty}^{\infty} \delta(x) dx = 1$ $\delta(y) = 0 \forall x \neq y$

$$\text{what is } \int_{-\infty}^{\infty} \delta(x-y) dx? \quad \int_{-\infty}^{\infty} \delta(x-y) f(x) dx = f(y)$$

We are convolving $f(y)$ against a delta and getting $f(x)$.

$$\hat{1} f(x) = \int_a^b \delta(x-y) f(y) dy = \sum_n |\phi_n\rangle \langle \phi_n| f \rangle$$

$$\sum_n \phi_n^*(y) \phi_n(x) = \delta(x-y) \quad \text{we assume } w(x) = 1 \quad \langle \phi_n| f \rangle$$

$$= \int_a^b \sum_n \phi_n^*(y) \phi_n(x) f(y) dy = \sum_n \phi_n(x) \int_a^b \phi_n^*(y) f(y) dy$$

$$\delta(x-y) = \sum_n w(x) \phi_n^*(y) \phi_n(x)$$

$$f(x) = \sum_n a_n \phi_n(x)$$

Linear operators

$A \Rightarrow$ Can be integer operators or differential operators or multiple by a factor.

$$A(af(x) + bg(x)) = aA[f(x)] + bA[g(x)]$$

$$(A+B)f(x) = A[f(x)] + B[f(x)]$$

$$A[f] = \int_a^b K(x,y) f(y) w(y) dy \quad \text{A kernel could be } \delta(x-y) \text{ or something like a gaussian}$$

Integral operators smooth functions

What is kernel dense estimate?

As we try to find the kernel associated with our given f functions?

• Operators can be algebra

$$\text{for example } p = -i \frac{d}{dx} \quad [x, p] = xp - px = -ix \frac{d}{dx} + i \frac{d}{dx} x = i \left[1 - x \frac{d}{dx} \right]$$

$$p = -i \frac{d}{dx} \quad x \quad [x, p]f = [xp - px]f = -ix \frac{df}{dx} + i \frac{d}{dx}(xf) = -ix \frac{df}{dx} + i \left[\frac{dx}{dx} f + x \frac{df}{dx} \right]$$

Do not simplify here.

$$\text{• Commutes with the algebra.} \quad \boxed{= i \frac{dx}{dx} f = if}$$

Finite dimensions

If $\hat{A}$ is Hermitian, then its eigenvalues are real and $\hat{v}$ form a complete basis.

Adjoint definition

For $f, g \in \mathcal{H}$ $A^\dagger$ is defined as:

$$\langle f | A^\dagger g \rangle = \langle A f | g \rangle$$

$$\langle f | (A^\dagger g) \rangle = (\langle f | A^\dagger) g \rangle$$

Assuming $w(x) = 1$ what is the purpose of $w(x)$?

$$\langle f | g \rangle = \int_a^b f(x) g(x) dx \quad \langle A^\dagger g | f \rangle = \langle g | A f \rangle$$

$$A = \frac{d}{dx} + V(x)$$

$$\begin{aligned} \langle A f | g \rangle &= \int_a^b \left(\frac{d}{dx} + V(x) \right) f(x) g(x) dx \\ &= \int_a^b \left(\frac{d}{dx} f(x) + V(x) f(x) \right) g(x) dx \\ &= \int_a^b \frac{d}{dx} f(x) g(x) + V(x) f(x) g(x) dx \\ &= \int_a^b \frac{d}{dx} f(x) g(x) + f(x) V(x) g(x) dx \end{aligned}$$

$$\begin{aligned} \int_a^b \frac{d}{dx} f(x) g(x) dx &= f(x) g(x) \Big|_a^b - \int_a^b f(x) \frac{d}{dx} g(x) dx \\ &= - \int_a^b f(x) \frac{d}{dx} g(x) + f(x) V(x) g(x) dx \\ &= - \int_a^b f(x) \left[\frac{d}{dx} + V(x) \right] g(x) dx = \langle f | A^\dagger g \rangle \\ &= \langle A f | g \rangle \end{aligned}$$

In a Hilbert space, functions must vanish at infinity. They are integrable.

• Why do functions have to be integrable?

• Does A have a complete basis of eigenfunctions?

- Domains of A and $A^\dagger$ must be the same

- Sturm-Liouville problems $\Rightarrow$ chapter 8 textbook!

$$A f = \lambda f \quad A = A^\dagger \quad \Rightarrow \text{what can we show?}$$

$$\langle f | A f \rangle = \langle A^\dagger f | f \rangle = \langle A f | f \rangle \quad \text{if } A = A^\dagger$$

$$\langle A f | f \rangle = \langle f | A^\dagger f \rangle = \langle f | A f \rangle \quad \text{if } A = A^\dagger$$

• A spectral theorem is required to show that A has a complete set of functions basis.

• Make sure your understanding agrees with this in finite space.

• Boundary terms are very important

Represent operators in a finite space:

$$A f = \sum_n a_n A \phi_n(x) \Rightarrow \sum_m \phi_m^*(x) \phi_m(x) \sum_n a_n A \phi_n(x)$$

$$\begin{aligned} \sum_m \phi_m^*(x) \phi_m(x) A \sum_n \phi_n^*(x) \phi_n(x) \\ = \sum_{m,n} [\phi_m(x) A \phi_n^*(x)] \phi_m^*(x) \phi_n(x) \end{aligned}$$

$$|g\rangle = |A f\rangle$$

$$= \sum_n |\phi_n\rangle \langle \phi_n | g \rangle = \sum_m |\phi_m\rangle \langle \phi_m | A | \sum_n |\phi_n\rangle \langle \phi_n | f \rangle$$

$$\text{Change basis: } |\phi_n'\rangle \quad A_{m'n'} = \langle \phi_m' | A | \phi_n' \rangle$$

$$= \int_a^b \phi_m'^*(x) A \phi_n'(x) w(x) dx$$

This is the projection into a new basis.

If we are concerned with basis transformation between orthonormal bases

then the basis does is under U .

We can always make certain f expressed in some basis