

Week 7: Mathematical physics

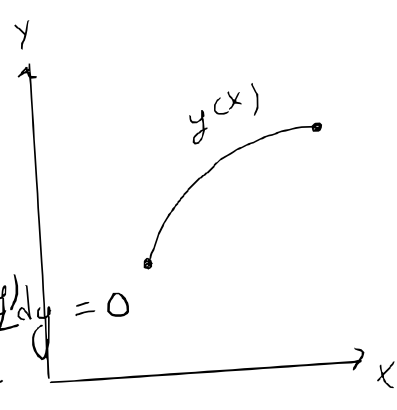
This week we will focus on first order equations

Separation of variables and ordinary differential equations.

$$\frac{dy}{dx} = F(x, y)$$

$$\Phi(x, y) = \text{const}$$

$$d\Phi(x, y) = \frac{\partial \Phi(x, y)}{\partial x} dx + \frac{\partial \Phi(x, y)}{\partial y} dy = 0$$



- what is a source differential equation?

Are all diff equations sourced?

We can always solve DEs of any order as long as they are linear and they have constant coefficients.

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi(x, y)}{\partial x} + \frac{\partial \Phi(x, y)}{\partial y} \frac{dy}{dx} = 0$$

$$\frac{\partial P}{\partial y} = - \frac{\partial \Phi}{\partial y \partial x}$$

$$\Rightarrow \frac{-\frac{\partial \Phi}{\partial x}}{\frac{\partial \Phi}{\partial y}} = \frac{dy}{dx} \Rightarrow \frac{-P(x, y)}{Q(x, y)}$$

If the path is given for the path

- Integrability condition $\Rightarrow$ Guarantees existence of source potential. $\Rightarrow$ Gauss path independent in the integral.

If true, $d\Phi = P dx + Q dy \rightarrow$ solved by integration

- we integrate in stages holding the dependent variable constant.

Example: $\frac{dy}{dx} = -\left(1 + \frac{y}{x}\right) \Rightarrow -\frac{x+y}{x}$

$$P(x, y) = x+y$$

$$Q(x, y) = x$$

$$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 1$$

$$d\Phi = P(x, y) dx + Q(x, y) dy \Rightarrow (x+y) dx + x dy$$

$$\Phi(x, y) = \int_0^x (x+y) dx = \left. \frac{1}{2} x^2 + xy \right|_{x_0}^x + x_0 y \Big|_{y_0}^y$$

$$\Rightarrow \Phi(x, y) = \frac{x^2}{2} + xy = C$$

$\Rightarrow$ The integral over x is at fixed y and y_0 integral $\rightarrow$ at initial x .

$$-\frac{1}{x} \left(\frac{x^2}{2} - C \right) = y \Rightarrow y(x) = \frac{C}{x} - \frac{x}{2} = \frac{2C - x^2}{2x}$$

$$xy = C - \frac{x^2}{2} \quad y = \frac{C}{x} - \frac{x}{2} = \frac{2C - x^2}{2x}$$

If the integrability condition is not satisfied, we can use an integrating factor.

$$\alpha(x, y) P(x, y) dx + \alpha(x, y) Q(x, y) dy = d\Phi$$

$$\frac{\partial \Phi}{\partial x} = \alpha(x, y) P(x, y)$$

Linear equations

Homogeneous equations have the same degree in all factors.

$$(x^2 y + x^3) dx + y^3 dy = 0$$

all cubes. $y = x u(x)$ and solve for u

Isobaric $y = x^m u(x)$ if we assign degrees of x, y and u we get $\rightarrow$

Linear first order ODE: $\rightarrow$ learn how to solve.

$$\frac{dy}{dx} + p(x)y = q(x)$$

$q(x) \neq 0$ inhomogeneous eq.

particular solution depends on forcing term.

particular solution depends on forcing term

Solution to general first order linear ODE.

$$y = y_p + C y_h$$

Change in α just change homogeneous solution.

2nd order linear ODEs (Homogeneous) Gauss form need for inhomogeneous eqn.

Gauss form need for 2nd order linear inhomogeneous eq.

Quantum Hydrogen atom or sphere symmetric in polar $\Rightarrow$ Legendre polynomials.

$\rightarrow$ Examine eigenvalues and see what you can tell about x or the radial wave to 0 or 1. Also check for elements of the problem.

+ Legendre polynomials
+ Associated Legendre polynomials
+ Bessel functions.

• Series expansion solutions to diff equations. 2nd order linear

• Classify equations.

+ Singular or ODEs.

+ $P(x), Q(x)$ are linear

$$P(x) = P(x_0) + \frac{d}{dx} (P(x-x_0))$$

• Regular singular points x_0

$$\lim_{x \rightarrow x_0} (x-x_0) P(x) \text{ exists} \rightarrow \lim_{x \rightarrow x_0} (x-x_0)^2 Q(x) \text{ exists}$$